§4.1 Vector Spaces

Goal: Create a generalization of $\mathbb{R}^n$ and its properties.

Defn: A vector space is a nonempty set V of objects called vectors equipped with two operations:

1) vector addition
2) scalar multiplication (by real numbers)

which satisfy the following axioms:

Assume u, v, and w are vectors in V and c, d scalars

1) $u+v$ is contained in V
2) $u+v = v+u$
3) $(u+v)+w = u+(v+w)$
4) There is a zero vector 0 in V such that $u+0=u$
5) There exists a vector $-u$ in V such that $u+(-u)=0$
6) $c \cdot u$ is contained in V
7) $c(u+v) = cu + cv$
8) $(c+d)u = cu + du$
9) $c(du) = (cd)u$
10) $1 \cdot u = u$

Examples

1) $\mathbb{R}^n$ is a vector space. We have already seen vectors in $\mathbb{R}^n$ satisfy these axioms

2) $\mathcal{M}_{mn}$, the set of all $m \times n$ matrices forms a vector space. (check this!)

<u>Defn</u>: Let $\mathbb{P}_n$ denote the set of all polynomials with real coefficients of degree at most n together with the zero polynomial $\mathbf{0}$.

<u>Claim</u>: $\mathbb{P}_n$ is a vector space. Let's check!

We verify axioms 1-10. Some are left as exercises.

Write
$$p(t) = a_0 + a_1 t + \cdots + a_n t^n$$
$$q(t) = b_0 + b_1 t + \cdots + b_n t^n$$

1) $p(t) + q(t) = (a_0+b_0) + (a_1+b_1)t + \cdots + (a_n+b_n)t^n$ is a polynomial of degree $\leq n$ so is in $\mathbb{P}_n$

2) exercise

3) exercise

4) $p(t) + \mathbf{0} = p(t)$

zero polynomial

5) $-p(t) = -a_0 - a_1 t - \cdots - a_n t^n$

$\Rightarrow p(t) + (-p(t)) = 0$

6) $c \cdot p(t) = c \cdot (a_0 + a_1 t + \cdots + a_n t^n) = ca_0 + (ca_1)t + \cdots + (ca_n)t^n$

is still a polynomial of degree $\leq n$

7) - 10) exercise.

<u>Defn</u>: A <u>subspace</u> of vector space V is a subset H of V such that:

1) The zero vector of V is contained in H.

2) For u, v in H $u+v$ is also in H.

3) For any u in H and scalar c, $c \cdot u$ is in H.

<u>Remarks</u>

1) Every subspace of a vector space is a vector space itself. (check this!)

2) $\{0\}$ is a subspace of every vector space.

Examples and Non-examples

1) $\mathbb{R}^2$ is <u>not</u> a subspace of $\mathbb{R}^3$ since its not even a subset. However, the subset of $\mathbb{R}^3$

$$H = \left\{ \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \;\middle|\; x, y \text{ real \#s} \right\}$$

is a subspace of $\mathbb{R}^3$, It's usually associated to $\mathbb{R}^2$ in a natural way.

2) $\mathbb{P}_2$ is a subspace of $\mathbb{P}_3$. (verify this!)
In general $\mathbb{P}_m$ is a subspace of $\mathbb{P}_n$ if $m \leq n$

The easiest way to create examples of subspaces is to use the notion of the <u>spanning set</u> from earlier,

<u>Defn</u>: If V is a vector space and $v_1, \ldots, v_s$ are vectors in V, then $\text{Span}\{v_1, \ldots, v_s\}$ is the set of all linear combinations of $v_1, \ldots, v_s$

$$\text{span}\{v_1, \ldots, v_s\} = \{c_1 v_1 + \cdots + c_s v_s \mid c_1 \ldots c_s \text{ real \#s}\}$$

If $v_1, \dots, v_s$ are vectors of vector space V, then $\text{span}\{v_1, \dots, v_s\}$ is a subspace of V

Check this!

Remarks

- We usually call $\text{span}\{v_1, \dots, v_s\}$ the subspace spanned by $v_1, \dots, v_s$
- If H is a subspace of V, we say H is spanned by $v_1, \dots, v_s$ if $H = \text{span}\{v_1, \dots, v_s\}$

Example

Let V be the vector space of all continuous real valued functions $\{f(x) : \mathbb{R} \to \mathbb{R}\}$. Here

check this is a vector space!

- $f+g = f(x) + g(x)$
- $c \cdot f = c \cdot f(x)$
- zero vector $f(x) = 0$ zero function

The following are subspaces of V

- $\text{span}\{\sin(x)\}$
- $\text{span}\{\cos(x), \sin(x)\}$
- $\text{span}\{e^x, x^2, \sin(3x)\}$
- etc.

subspaces like this will appear frequently in differential equations